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# The alternating polynomials and their relation with the spectra and conditional diameters of graphs[1]

M.A. Fiol*, E. Garriga, J.L.A. Yebra

*Departament de Matemàtica Aplicada i Telemàtica, c/. Gran Capità, s/n, Mòdul C3, Campus Nord, Universitat Politècnica de Catalunya, 08034-Barcelona, Spain*

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**Abstract**

Given a graph $\Gamma$ on $n = |V\Gamma|$ vertices, the distance between two subgraphs $\Gamma_1, \Gamma_2 \subset \Gamma$, denoted by $\partial(\Gamma_1, \Gamma_2)$, is the minimum among the distances between vertices of $\Gamma_1$ and $\Gamma_2$. For some integers $1 \leqslant s,\ t \leqslant n$, the conditional $(s,t)$*-diameter* of $\Gamma$ is then defined as $D_{(s,t)} = \max_{\Gamma_1,\Gamma_2 \subset \Gamma}\{\partial(\Gamma_1, \Gamma_2)\colon\ |V\Gamma_1| = s,\ |V\Gamma_2| = t\}$. Let $\Gamma$ have distinct eigenvalues $\lambda > \lambda_1 > \lambda_2 > \cdots > \lambda_d$. For every $k = 0, 1, \ldots, d-1$, the $k$*-alternating* polynomial $P_k$ is defined to be the polynomial of degree $k$ and norm $\|P_k\|_\infty = \max_{1 \leqslant l \leqslant d}\{|P_k(\lambda_l)|\} = 1$ that attains maximum value at $\lambda$. These polynomials, which may be thought of as the discrete version of the Chebychev ones, were recently used by the authors to bound the (standard) diameter $D \equiv D_{(1,1)}$ of $\Gamma$ in terms of its eigenvalues. In this work we derive similar results for conditional diameters. For instance, it is shown that

$$P_k(\lambda) > \frac{\|\boldsymbol{v}\|^2}{s} - 1 \ \Rightarrow\ D_{(s,s)} \leqslant k,$$

where $\boldsymbol{v}$ is the (positive) eigenvector associated to $\lambda$ with minimum component 1. Similar results are given for locally regular digraphs by using the Laplacian spectrum. Some applications to the study of other parameters, such as the connectivity of $\Gamma$, are also discussed.

## 1. Introduction

Throughout this paper, otherwise stated, $\Gamma = (V, E)$ denotes a (simple and finite) connected graph, with vertex set $V = V\Gamma$, $|V| = n$, and edge set $E = E\Gamma$. For any vertex $e_i \in V$, $\Gamma(e_i)$ denotes the set of vertices adjacent to $e_i$, and $\delta_i = |\Gamma(e_i)|$ denotes its *degree*. Then, $\Gamma$ is ($\delta$-)*regular* if $\delta_i = \delta$ for all $1 \leqslant i \leqslant n$. The *distance* between two vertices $e_i$ and $e_j$ will be denoted by $\partial(e_i, e_j)$, and $D = D(\Gamma)$ stands for the *diameter*.

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* Corresponding author. E-mail: fiol@mat.upc.es, egarriga@mat.upc.es, yebra@mat.upc.es.


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Given $e_i \in V$ and any integer $\rho$, $0 \leqslant \rho \leqslant D$, $\Gamma_\rho(e_i)$ denotes the set of vertices which are at distance $\rho$ from $e_i$.

Given two (not necessarily connected) subgraphs $\Gamma_1, \Gamma_2$, of a graph $\Gamma$, the *distance* from $\Gamma_1$ to $\Gamma_2$ is defined as

$$\partial(\Gamma_1, \Gamma_2) = \min\{\partial(e_i, e_j)\colon e_i \in V\Gamma_1, e_j \in V\Gamma_2\}.$$

The distance between two subsets of vertices $V_1, V_2 \subset V$, denoted by $\partial(V_1, V_2)$, is defined analogously and, clearly, $\partial(\Gamma_1, \Gamma_2) = \partial(V\Gamma_1, V\Gamma_2)$. If one of the subgraphs or vertex sets, say $\Gamma_1$ or $V_1$, consists of a single vertex $e_i$, we simply write $\partial(e_i, G_2)$ or $\partial(e_i, V_2)$ respectively. Given $S \subset V$, we will denote by $\Gamma_\rho(S)$ the set of vertices which are at distance $\rho$ from $S$, that is, $\Gamma_\rho(S) = \{e_i \in V\colon\ \partial(e_i, S) = \rho\}$. Thus, $\Gamma(e_i) = \Gamma_1(e_i) \equiv \Gamma_1(\{e_i\})$.

Given a property $\mathscr{P}$ of a pair $(\Gamma_1, \Gamma_2)$ of subgraphs of $\Gamma$, the so-called *conditional diameter* or *$\mathscr{P}$-diameter*, introduced in [2], measures the maximum distance among subgraphs satisfying $\mathscr{P}$. That is,

$$D_{\mathscr{P}} = D(\Gamma, \mathscr{P}) = \max_{\Gamma_1, \Gamma_2 \subset \Gamma} \{\partial(\Gamma_1, \Gamma_2)\colon (\Gamma_1, \Gamma_2) \text{ satisfy } \mathscr{P}\}.$$

So the study of this parameter could be of some interest in the design of interconnection networks when we need to minimize the communication delays between the clusters modeled by such subgraphs. For instance, if $\mathscr{P}$ is the property of $\Gamma_i$, $i = 1, 2$, being trivial (that is, isolated vertices) the conditional diameter $D_{\mathscr{P}}$ coincides with the standard diameter $D$. Another example, considered in this paper, is the *$(s, t)$-diameter* $D_{(s,t)}$, obtained when $\mathscr{P}$ requires that $|V\Gamma_1| = s$ and $|V\Gamma_2| = t$ for some integers $1 \leqslant s, t \leqslant n$.

Let $A$ be the *adjacency matrix* of $\Gamma$ that has $d+1$ distinct eigenvalues $\lambda_0 > \lambda_1 > \cdots > \lambda_d$. Because of its special role, the largest eigenvalue $\lambda_0$ is also denoted by $\lambda$. As a consequence of the theorem of Perron–Frobenius, $\lambda$ is simple and positive, with positive eigenvector $\boldsymbol{v} = (v_1, v_2, \ldots, v_n)^{\mathrm{T}}$. As usual, we identify $A$ with an endomorphism of the 'vertex-space' of $\Gamma$, $\ell^2(V)$ which, for any given indexing of the vertices, is isomorphic to $\mathbb{R}^n$. Thus, for a given ordering of its vertices, we only distinguish between a vertex $e_i$ and the corresponding vector $\boldsymbol{e}_i$ of the canonical base of $\mathbb{R}^n$ by the bold type used.

Recently, some results relating the diameter of a graph and its different eigenvalues have been given. Thus, following the works of Chung [4], Chung et al. [5], Delorme and Solé [6], Mohar [12], Quenell [13], and Sarnak [14], the authors showed that, if $\lambda > \lambda_1 > \cdots > \lambda_d$ are the distinct eigenvalues of a graph $\Gamma$ with diameter $D(\Gamma)$, then

$$P_k(\lambda) > \|\boldsymbol{v}\|^2 - 1 \Rightarrow D(\Gamma) \leqslant k \tag{1}$$

where $\boldsymbol{v}$ is the positive eigenvector corresponding to $\lambda$ with minimum component 1, and $P_k$ is the $k$-alternating polynomial on the mesh $\mathscr{M} = \{\lambda_1 > \cdots > \lambda_d\}$, defined in the next section.

In the case of regular graphs, we have $\boldsymbol{v} = \boldsymbol{j}$, the all-1 vector, and hence the result (1) simplifies to

$$P_k(\lambda) > n - 1 \Rightarrow D(\Gamma) \leqslant k. \tag{2}$$

This paper gives similar results by considering some conditional diameters.

In order to judge the accuracy of (1), the authors explored those graphs satisfying $P_k(\lambda) = \|\boldsymbol{v}\|^2 - 1$, which were called *k-boundary graphs* in [7]. In particular, from Eq. (5) given below, $(d-1)$-boundary graphs, called *boundary graphs* for short, are characterized by

$$\sum_{l=0}^{d} \frac{\pi_0}{\pi_l} = \|\boldsymbol{v}\|^2 \quad \text{where } \pi_l = \prod_{j=0 (j \neq l)}^{d} |\lambda_l - \lambda_j| \ (0 \leqslant l \leqslant d). \tag{3}$$

Some basic properties of these graphs were discussed in [8, 9], whereas [10] dealt with the cases $k < d - 1$. Also, some constructions of boundary graphs were presented, both in the regular and nonregular case. In particular, it was shown that there are boundary graphs with diameter $D \leqslant d - 1$, but there are also such graphs with spectrally maximum diameter $D = d$, which we call *extremal.* In [9], it was shown that a graph is an extremal and *diametral* (i.e. all vertices with maximum eccentricity) boundary graph if and only if it is a 2-antipodal distance-regular graph.

Upper bounds on the diameter of a (not necessarily regular) (di)graph, in terms of the eigenvalues of its Laplacian matrix have also been derived by Alon and Milman [1], Mohar [12], and Chung et al. [5]. Recall that the *Laplacian matrix* of a graph $\Gamma$, denoted by $\boldsymbol{Q} = \boldsymbol{Q}(\Gamma)$, is defined as $\boldsymbol{Q} = \boldsymbol{D} - \boldsymbol{A}$, where $\boldsymbol{D} = \operatorname{diag}(\delta_1, \delta_2, \ldots, \delta_n)$. Alternatively, the Laplacian matrix can be defined by $\boldsymbol{Q} = \boldsymbol{C}\boldsymbol{C}^{\mathrm{T}}$, where $\boldsymbol{C}$ is the incidence matrix of an orientation of $\Gamma$ (see [3]). A comprehensive survey about the properties and applications of the Laplacian matrix can be found in Mohar [11]. For instance, $\boldsymbol{Q}$ has eigenvalues $\mu_0 = 0 < \mu_1 < \cdots < \mu_d$, and the (simple) eigenvalue 0 has eigenvector $\boldsymbol{j}$. Moreover, notice that $\boldsymbol{Q}$ can be seen as the adjacency matrix of a weighted pseudograph, obtained from $\Gamma$ by giving weight $-1$ to its edges and adding a loop with weight $\delta_i$ on each vertex $e_i$. Therefore, as when using $\boldsymbol{A}$, if $(P(\boldsymbol{Q}))_{ij} \neq 0$ for a polynomial $P$ of degree $k$, there must be some path of length $\leqslant k$ in $\Gamma$ between vertices $e_i$ and $e_j$. The above fact leads to the analogue of (1) and (2), see [7]: Let $\Gamma$ be a graph with Laplacian matrix eigenvalues $\mu_0 = 0 < \mu_1 < \cdots < \mu_d$. Then,

$$Q_k(0) > n - 1 \Rightarrow D(\Gamma) \leqslant k, \tag{4}$$

where $Q_k$ is now the alternating polynomial computed on the nonzero eigenvalues of $\boldsymbol{Q}$. Since the Laplacian matrix can also be defined for a *locally regular* digraph (that is $\delta_i^+ = \delta_i^-$ for all $e_i \in V$), and some of its properties still hold, a similar result holds for such digraphs.

## 2. The alternating polynomials

Let $\mathscr{M} = \{\lambda_1 > \cdots > \lambda_d\}$ be a mesh of $d$ real numbers. For any $k = 0, 1, \ldots, d-1$ let $P_k$ denote the *k-alternating polynomial* associated to $\mathscr{M}$. That is, the polynomial of

$\mathbb{R}_k[x]$ with norm $\|P_k\|_\infty \leqslant 1$, defined by

$$P_k(\lambda) = \sup\{p(\lambda): p \in \mathbb{R}_k[x], \|p\|_\infty \leqslant 1\} \quad \text{where } \|p\|_\infty = \max_{1 \leqslant i \leqslant d} \{|p(\lambda_i)|\}$$

and $\lambda$ is any real number greater than $\lambda_1$. In [7], it was shown that, for any $k = 0, 1, \dots, d-1$,

- There is a unique $P_k$ which, moreover, is independent of the value of $\lambda (> \lambda_1)$;
- $P_k$ has degree $k$;
- All the roots of $P_k$ are real, simple, and within the interval $(\lambda_d, \lambda_1)$;
- $P_0(\lambda) = 1 < P_1(\lambda) < P_2(\lambda) < \cdots < P_{d-1}(\lambda)$;
- Let $p \in \mathscr{B}_k = \{p \in \mathbb{R}_k[x]: \|p\|_\infty \leqslant 1\}$. Then $p = P_k$ if and only if there exist a (not necessarily unique) submesh $\mathscr{N}_k = \{\mu_1 > \mu_2 > \cdots > \mu_{k+1}\} \subseteq \mathscr{M}$, with $\mu_1 = \lambda_1$ and $\mu_{k+1} = \lambda_d$, such that $p(\mu_l) = (-1)^{l+1}$. Hence, in particular, $P_k(\lambda_1) = 1$ and $P_k(\lambda_d) = (-1)^k$.

In other words, $P_k$ takes $k+1$ *alternating* values $\pm 1$ at the mesh points, as does the Chebychev polynomial $T_k$ in $[-1, +1]$. Moreover, the submesh $\mathscr{N}_k$ uniquely characterizes $P_k$, and Lagrange interpolation at $\mu_1, \dots, \mu_{k+1}$ gives:

$$P_k(\lambda) = \sum_{l=1}^{k+1} \frac{\pi_0}{\pi_l} \quad \text{where } \pi_l = \prod_{j=0 (j \neq l)}^{k+1} |\mu_l - \mu_j| \ (0 \leqslant l \leqslant k+1),\ \mu_0 = \lambda. \tag{5}$$

The $k$-alternating polynomials of degree 1, 2 and $d-1$ can be made explicit since, as it is easily checked, their corresponding submeshes are (see also [7]):

- $\mathscr{N}_1 = \{\lambda_1 > \lambda_d\}$;
- $\mathscr{N}_2 = \{\lambda_1 > \mu_2 > \lambda_d\}$, where $\mu_2 \in \mathscr{M}$ is a closest point to $(\lambda_1 + \lambda_d)/2$;
- $\mathscr{N}_{d-1} = \mathscr{M}$ (hence, $P_{d-1}(\lambda_l) = (-1)^{l+1}, l = 1, 2, \dots, d$).

To the knowledge of the authors, there are no explicit expressions for the other alternating polynomials. However, in the same paper it was shown that the problem of finding $P_k$ for any $k$ can be translated to the following linear programming problem, which can be solved by using the simplex method. Let $P_k \in \mathbb{R}_k[x]$ be the polynomial defined by $P_k(\lambda_l) = x_l$, $1 \leqslant l \leqslant d$, where the vector $(x_1, x_2, \dots, x_d)$ is a solution of the problem:

$$\begin{array}{lll} \text{maximize} & x_0 & \\ \text{with constraints} & f[\lambda_0, \lambda_1, \dots, \lambda_m] = 0, & m = k+1, \dots, d. \\ & x_l \leqslant 1, \quad x_l \geqslant -1, & l = 1, 2, \dots, d. \end{array}$$

where $f[\lambda_0, \lambda_1, \dots, \lambda_m]$ denote the $m$th divided differences of Newton interpolation, recursively defined by

$$f[\lambda_0, \lambda_1, \dots, \lambda_l] = \frac{f[\lambda_1, \dots, \lambda_l] - f[\lambda_0, \dots, \lambda_{l-1}]}{\lambda_l - \lambda_0},$$

starting from $f[\lambda_l] = P_k(\lambda_l) = x_l$, $0 \leqslant l \leqslant d$.

For example, when considering the points:

$$\lambda_1 = 4, \qquad \lambda_2 = 2, \qquad \lambda_3 = -1, \qquad \lambda_4 = -3, \qquad \lambda_5 = -5$$

the corresponding $k$-alternating polynomials, $1 \leqslant k \leqslant 4$, and their values at the mesh points $(\lambda_5, \lambda_4, \lambda_3, \lambda_2, \lambda_1)$, and at $\lambda = 6$ (the different eigenvalues of the *odd graph* $O_6$, see [3]), are

- $P_4(x) = \frac{1}{42}(x^4 + 2x^3 - 19x^2 - 20x + 42)$, $(1, -1, 1, -1, 1)$, $23$;
- $P_3(x) = \frac{1}{45}(2x^3 + 3x^2 - 29x - 15)$, $(-1, 1, \frac{1}{3}, -1, 1)$, $\frac{39}{5}$;
- $P_2(x) = \frac{1}{10}(x^2 + x - 10)$, $(1, -\frac{2}{5}, -1, -\frac{2}{5}, 1)$, $\frac{16}{5}$;
- $P_1(x) = \frac{1}{9}(2x + 1)$, $(-1, -\frac{5}{9}, -\frac{1}{9}, \frac{5}{9}, 1)$, $\frac{13}{9}$.

The alternating polynomials can also be defined on the complex plane. Now, $\mathcal{M}$ is a set of $d$ complex numbers $\lambda_1, \ldots, \lambda_d$, contained in some closed circle $\mathscr{E}(c, r) = \{z \in \mathbb{C}: |z - c| \leqslant r\}$. Let $\mathbb{C}_k[z]$ denote the vector space of polynomials over $\mathbb{C}$, with degree at most $k$. We consider the norm $\|p\|_\infty = \max_{1 \leqslant l \leqslant d}\{|p(\lambda_l)|\}$, and the closed unit ball $\mathscr{B}_k = \{p \in \mathbb{C}_k[z]: \|p\|_\infty \leqslant 1\}$. Let $\lambda \notin \mathscr{E}(c, r)$. The *(complex) k-alternating polynomial* on $\mathcal{M}$ is then a point of $\mathscr{B}_k$ where the function $\Psi: p \to p(\lambda)$ takes maximum real value.

With this definition we have, as in the real case, $P_k(\lambda) \geqslant \max_{p+q=k}\{P_p(\lambda)P_q(\lambda)\}$ $(1 \leqslant k \leqslant d - 1)$, since $P_1(\lambda) \geqslant (1/r)|\lambda - c| > 1$. So, we have $\operatorname{grd} P_k = k$ and $P_1(\lambda) < P_2(\lambda) < \cdots < P_{d-1}(\lambda)$ again. Also, the uniqueness of the complex alternating polynomial can be basically proved as in the real case. Furthermore, for $k = d - 1$ we get $P_{d-1}(\lambda) = \sum_{l=1}^{d} \pi_0/\pi_l$, with $\pi_l = \prod_{j=0(j \neq l)}^{d} |\lambda_l - \lambda_j|$, $0 \leqslant l \leqslant d$, $\lambda_0 = \lambda$.

## 3. Conditional diameters and eigenvalues

In this section we use the alternating polynomials for deriving some results about the connection between the (standard or Laplacian) spectrum of a graph and some of its parameters, mainly conditional diameters.

### *3.1. Adjacency matrix spectrum and (s, t)-diameters*

The following theorem is a generalization, for conditional $(s, t)$-diameters, of the results in [7]. The proof is mainly based on the positiveness of the eigenvector $\boldsymbol{v}$, associated to $\lambda$. Recall also that, by the spectral theorem, the vector space $\ell^2(V)$ has got an orthogonal basis consisting of eigenvectors of $A$.

**Theorem 3.1.** *Let $\Gamma = (V, E)$ be a graph with eigenvalues $\lambda > \lambda_1 > \cdots > \lambda_d$, and let $P_k$ denote the k-alternating polynomial on the mesh $\{\lambda_1 > \cdots > \lambda_d\}$. Let $\boldsymbol{v}$ be the eigenvector associated to $\lambda$ with minimum component* 1. *Then,*

$$P_k(\lambda) > \sqrt{\left(\frac{\|\boldsymbol{v}\|^2}{s} - 1\right)\left(\frac{\|\boldsymbol{v}\|^2}{t} - 1\right)} \;\Rightarrow\; D_{(s,t)}(\Gamma) \leqslant k. \tag{6}$$

**Proof.** Let $A$ be the adjacency matrix of $\Gamma$, and represent by $\boldsymbol{e}_i$ the $i$th coordinate vector. Consider two disjoint generic subsets $V_i, V_j \subset V$, with cardinalities $s$ and $t$ respectively, which are represented by the vectors $\boldsymbol{f}_i = \sum_{g=1}^{s} \boldsymbol{e}_{i_g}$ and $\boldsymbol{f}_j = \sum_{h=1}^{t} \boldsymbol{e}_{j_h}$. We want to show that, under the stated hypothesis, the distance between some vertex of

$V_i$ and some vertex of $V_j$ does not exceed $k$. But, clearly, this is the case when $\sum_{g=1}^{s}\sum_{h=1}^{t}(P_k(A))_{i_g j_h} > 0$

Thus, by using the decompositions

$$f_i = \frac{\langle f_i, v\rangle}{\|v\|^2}v + z_i = \frac{\sum_{g=1}^{s} v_{i_g}}{\|v\|^2}v + z_i; \qquad f_j = \frac{\sum_{h=1}^{t} v_{j_h}}{\|v\|^2}v + z_j,$$

where $z_i, z_j \in v^{\perp}$, we get

$$\sum_{g=1}^{s}\sum_{h=1}^{t}(P_k(A))_{i_g j_h} = \langle P_k(A)f_i, f_j\rangle = \frac{\sum_{g=1}^{s} v_{i_g}\sum_{h=1}^{t} v_{j_h}}{\|v\|^2}P_k(\lambda) + \langle P_k(A)z_i, z_j\rangle.$$

Moreover,

$$\|z_i\|^2 = \|f_i\|^2 - \frac{(\sum_{g=1}^{s} v_{i_g})^2}{\|v\|^2} \leqslant s - \frac{s^2}{\|v\|^2}; \quad \|z_j\|^2 \leqslant t - \frac{t^2}{\|v\|^2}$$

because $v$ has all components $\geqslant 1$. Then,

$$|\langle P_k(A)z_i, z_j\rangle| \leqslant \|P_k(A)z_i\|\,\|z_j\| \leqslant \|P_k\|_\infty \|z_i\|\,\|z_j\|$$
$$\leqslant \frac{st}{\|v\|^2}\sqrt{\left(\frac{\|v\|^2}{s} - 1\right)\left(\frac{\|v\|^2}{t} - 1\right)}$$

since $\|P_k(A)|_{v^\perp}\| = \|P_k\|_\infty = 1$. Therefore,

$$\sum_{g=1}^{s}\sum_{h=1}^{t}(P_k(A))_{i_g j_h} \geqslant \frac{st}{\|v\|^2}\left(P_k(\lambda) - \sqrt{\left(\frac{\|v\|^2}{s} - 1\right)\left(\frac{\|v\|^2}{t} - 1\right)}\right) > 0 \tag{7}$$

as claimed. □

In particular, for $s = t$ we obtain the following result:

**Corollary 3.2.** *Let $\Gamma$ be a graph with eigenvalues $\lambda > \lambda_1 > \cdots > \lambda_d$, and let $P_k$ be the $k$-alternating polynomial on $\{\lambda_1 > \cdots > \lambda_d\}$. Let $v$ be the eigenvector associated to $\lambda$ with minimum component* 1. *Then,*

$$P_k(\lambda) > \frac{\|v\|^2}{s} - 1 \Rightarrow D_{(s,s)}(\Gamma) \leqslant k. \tag{8}$$

In particular, by using the results of Section 2 characterizing the alternating polynomials $P_2$ and $P_{d-1}$, we can state the following result.

**Corollary 3.3.** *Let $\Gamma$ be a graph with eigenvalues $\lambda_0(=\mu_0=\lambda)$ and $\mathcal{M} = \{\lambda_1 > \cdots > \lambda_d\}$, $d \geqslant 3$; and let $\mathcal{N}_2 = \{\mu_1(=\lambda_1) > \mu_2 > \mu_3(=\lambda_d)\} \subset \mathcal{M}$ denote* *the submesh where $P_2$ takes alternating values* $\pm 1$. *Then,*

(a) $\sum_{l=0}^{d}(\pi_0/\pi_l) > (\|\boldsymbol{v}\|^2/s) \Rightarrow D_{(s,s)}(\Gamma) \leqslant d-1$,

$$\text{where } \pi_l = \prod_{j=0(j\neq l)}^{d} |\lambda_l - \lambda_j| \quad (0 \leqslant l \leqslant d);$$

(b) $\sum_{l=0}^{3}(\pi_0/\pi_l) > (\|\boldsymbol{v}\|^2/s) \Rightarrow D_{(s,s)}(\Gamma) \leqslant 2$,

$$\text{where } \pi_l = \prod_{j=0(j\neq l)}^{3} |\mu_l - \mu_j| \quad (0 \leqslant l \leqslant 3).$$

Corollary 3.2 can also be used to derive a result on the diameter of the line graph $L\Gamma$. Recall that in the line graph $L\Gamma$ of a graph $\Gamma$, each vertex represents an edge of $\Gamma$, and two vertices are adjacent iff the corresponding edges are. Then, since each edge can be seen as a subgraph on 2 vertices, the diameter of $L\Gamma$ satisfies $D(L\Gamma) \equiv D_{(1,1)}(L\Gamma) \leqslant D_{(2,2)}(\Gamma) + 1$.

**Corollary 3.4.** *Let $\Gamma$ be a graph as above, and consider its line graph $L\Gamma$. Then,*

$$P_k(\lambda) > \frac{\|\boldsymbol{v}\|^2}{2} - 1 \Rightarrow D(L\Gamma) \leqslant k+1. \tag{9}$$

Since for regular graphs $\boldsymbol{v} = \boldsymbol{j}$, Theorem 3.1 leads to the following result, which was also implicitly proved by Van Dam and Haemers in [15] (using a generic polynomial and without mention to the conditional diameters).

**Corollary 3.5.** *Let $\Gamma$ be a regular graph on n vertices, and with eigenvalues denoted as above. Then,*

(a) $P_k(\lambda) > \sqrt{(n/s-1)(n/t-1)} \Rightarrow D_{(s,t)}(\Gamma) \leqslant k \quad (1 \leqslant s,t \leqslant n)$;
(b) $P_k(\lambda) > n/s - 1 \Rightarrow D_{(s,s)}(\Gamma) \leqslant k$;
(c) $\sum_{l=0}^{d}(\pi_0/\pi_l) > (n/s) \Rightarrow D_{(s,s)}(\Gamma) \leqslant d-1$,
*where* $\pi_l = \prod_{j=0(j\neq l)}^{d} |\lambda_l - \lambda_j|\ (0 \leqslant l \leqslant d)$;
(d) $\sum_{l=0}^{3}(\pi_0/\pi_l) > (n/s) \Rightarrow D_{(s,s)}(\Gamma) \leqslant 2$,
*where* $\pi_l = \prod_{j=0(j\neq l)}^{3} |\mu_l - \mu_j|\ (0 \leqslant l \leqslant 3)$.

To illustrate Corollary 3.5, take first $s \leqslant n/2$ and $t = n - s$. Then, the condition in case (a) turns out to the $P_k(\lambda) > 1$, which is clearly satisfied with the alternating polynomial $P_1$. Hence, we conclude that $D_{(s,n-s)}(\Gamma) = 1$, as expected since $\Gamma$ is connected. A more interesting conclusion is reached when considering an *r-antipodal distance-regular* graph $\Gamma$ (see, for instance, [3]). In [8], the authors showed that the $(d-1)$-alternating polynomial of a such graph with $n$ vertices satisfies $P_{d-1}(\lambda) = (2/r)n - 1$. But this is precisely the value of $n/s - 1$ when $r$ is even and $s = r/2$. So, in a sense, we can say that $\Gamma$ is a $(d-1)$-boundary graph with respect to the diameter $D_{(r/2,r/2)}$ and, consequently, we cannot infer that $D_{(r/2,r/2)}(\Gamma) \leqslant d-1$. In fact, since $\Gamma$ is $r$-antipodal, given any $e_i \in V$, the set $\{e_i\} \cup \Gamma_D(e_i)$ has $r$ vertices which are mutually at distance $d$, and hence $D_{(r/2,r/2)}(\Gamma) = d$.

*3.2. Some applications*

We are now going to give some other consequences of Theorem 3.1 involving other parameters of a graph. To begin with, we derive an upper bound for the number $\sigma_k(S)$ of vertices which are at distance $\geqslant k$ from a given subset $S \subset V$, that is, $\sigma_k(S) = \sum_{\rho=k}^{D} |\Gamma_\rho(S)|$.

**Proposition 3.6.** *Let $\Gamma = (V, E)$ be a graph with diameter $D$. Then, given any subset $S \subset V$ of cardinality $|S| = s$, the number $\sigma_k(S)$ of vertices which are at distance $\geqslant k$ from $S$ is upper-bounded by*

$$\sigma_k(S) \leqslant \left\lfloor \frac{\|\boldsymbol{v}\|^2(\|\boldsymbol{v}\|^2 - s)}{s(P_{k-1}^2(\lambda) - 1) + \|\boldsymbol{v}\|^2} \right\rfloor \quad (1 \leqslant k \leqslant D). \tag{10}$$

**Proof.** Since clearly $D_{(s,\sigma_k(S))}(\Gamma) > k - 1$, the converse of Theorem 3.1 gives

$$P_{k-1}^2(\lambda) \leqslant \left(\frac{\|\boldsymbol{v}\|^2}{s} - 1\right)\left(\frac{\|\boldsymbol{v}\|^2}{\sigma_k(S)} - 1\right).$$

So, solving for $\sigma_k(S)$, and considering that it is an integer, we get (10). □

Note that, whenever $k = 1$, we have $P_0 = 1$, and (10) particularizes to the trivial bound $\sigma_1(S) \leqslant \|\boldsymbol{v}\|^2 - s$. Furthermore, when $\Gamma$ is regular, we get the following result which improves that given in [15] by using Chebychev polynomials.

**Corollary 3.7.** *Let $\Gamma = (V, E)$ be a regular graph with $n$ vertices and diameter $D$, and consider $S \subset V$, $\sigma_k(S)$ as above. Then,*

$$\sigma_k(S) \leqslant \left\lfloor \frac{n(n-s)}{s(P_{k-1}^2(\lambda) - 1) + n} \right\rfloor \quad (1 \leqslant k \leqslant D). \tag{11}$$

When $s = 1$ and $\Gamma$ has spectrally maximum diameter, the above corollary yields:

**Corollary 3.8.** *Let $\Gamma = (V, E)$ be an extremal regular graph with $n$ vertices and diameter $D(= d)$. Let the corresponding $(d-1)$-alternating polynomial be simply denoted by $P$. Then, for any given vertex $e_i \in V$, number $\sigma_d(e_i)$ of 'diametral vertices' with $e_i$ satisfies*

$$\sigma_d(e_i) \leqslant \left\lfloor \frac{n(n-1)}{n - 1 + P^2(\lambda)} \right\rfloor \tag{12}$$

*where $P(\lambda)$ is given by* (5).

For instance, the Petersen graph has eigenvalues 3, 1, − 2, and a simple computation gives $P(3) = \frac{7}{3}$. Thus, the result (12) gives the optimal bound

$$\sigma_2(e_i) \leqslant \left\lfloor \frac{90}{9} + P^2(3) \right\rfloor = 6.$$

Another interesting consequence of Proposition 3.6 gives a lower bound for the connectivity of a graph. First, recall that the *connectivity* (or *vertex-connectivity*) of $\Gamma$, usually denoted by $\kappa = \kappa(\Gamma)$, is the smallest number of vertices whose deletion results in a graph that is either nonconnected or trivial. In [2] it was shown that if $D_{(2,2)} = 1$, then $\Gamma$ is *maximally connected*, that is, its connectivity attains its maximum value $\kappa = \delta = \min_{1 \leqslant i \leqslant n} \delta_i$. For other values of $D_{(2,2)}$ we can give the following result.

**Proposition 3.9.** *Let $\Gamma = (V, E)$ be a graph with minimum degree $\delta$ and conditional diameter $D_{(2,2)} > 1$. Then, its connectivity $\kappa$ satisfies*

$$\kappa \geqslant \min\left\{\delta, n - 2 - \left\lfloor \frac{\|\boldsymbol{v}\|^2(\|\boldsymbol{v}\|^2 - 2)}{2(P_1^2(\lambda) - 1) + \|\boldsymbol{v}\|^2} \right\rfloor\right\}. \tag{13}$$

**Proof.** Notice first that, if a proper subset of $V$, say $S \neq \emptyset$, satisfies $S \cup \Gamma_1(S) \neq V$, then $\Gamma_1(S)$ is a cutset of $\Gamma$. Thus, the connectivity of $\Gamma$ is $\kappa = \min\{|\Gamma_1(S)|: 1 \leqslant s \leqslant \lfloor n/2 \rfloor\}$. Moreover, $|\Gamma_1(S)| = n - s - \sigma_2(S)$ and, by (10),

$$\sigma_2(S) \leqslant \phi(s) = \frac{\|\boldsymbol{v}\|^2(\|\boldsymbol{v}\|^2 - s)}{s(P_1^2(\lambda) - 1) + \|\boldsymbol{v}\|^2}.$$

If the minimum of $|\Gamma_1(S)|$ is attained with $s = 1$ we clearly have $\kappa = \delta$. Otherwise, when $2 \leqslant s \leqslant \lfloor n/2 \rfloor$, we claim that the maximum of the function $\psi(s) = s + \phi(s)$ is attained for $s = 2$, which gives the desired bound. Indeed, for $s > 0$, $\psi(s)$ attains only a local minimum at the point $s_0 = \|\boldsymbol{v}\|^2/(P_1(\lambda) + 1)$. Moreover, $s_0 \in [2, \|\boldsymbol{v}\|^2/2)$ since $P_1(\lambda) > 1$, and $s_0 < 2$ would give $P_1(\lambda) > \|\boldsymbol{v}\|^2/2 - 1 \Rightarrow D_{(2,2)} = 1$ contradicting the hypothesis. Hence, $\max_{s \in [2, \lfloor n/2 \rfloor]} \psi(s)$ is attained at one of the extremes of the interval, and simple reasoning shows that this must be 2. $\square$

When $\Gamma$ is a regular graph, the expression on the right-hand side of (13) can be slightly simplified, as the next result shows.

**Corollary 3.10.** *Let $\Gamma = (V, E)$ be a $\delta$-regular graph with conditional diameter $D_{(2,2)} > 1$. Then, its connectivity $\kappa$ satisfies*

$$\kappa \geqslant \min\left\{\delta, \left\lceil \frac{2(P_1^2(\lambda) - 1)(n - 2)}{2(P_1^2(\lambda) - 1) + n} \right\rceil\right\}. \tag{14}$$

In particular, notice that

$$\left\lceil \frac{2(P_1^2(\lambda) - 1)(n - 2)}{2(P_1^2(\lambda) - 1) + n} \right\rceil \geqslant \delta$$

is a sufficient condition for $\Gamma$ to be maximally connected. By way of example, consider the odd graph $O_4$, which is a 4-regular graph on $n = 35$ vertices and eigenvalues $\lambda = 4$,

$\lambda_1 = 2$, $\lambda_2 = -1$ and $\lambda_3 = -3$. Then, $P_1(4) = \frac{9}{5}$ and (14) gives $\kappa \geqslant 4$, so that $O_4$ is maximally connected.

### 3.3. Laplacian matrix spectrum and (s, t)-diameters

Reasonings similar to those used in Section 3.1 lead to analogous conditions, involving the spectrum of the Laplacian matrix $\boldsymbol{Q}$, to bound the conditional diameters of a (not necessarily regular) graph. The only significant change is that we must now consider the appropriate alternating polynomial, according to the characteristics of such spectrum. Thus, let $Q_k$ be the polynomial defined a $Q_k = (-1)^k P_k$ which, among the polynomials of $\mathscr{B}_k$, attains maximum value at any point on the left of $\mu_1, \ldots, \mu_d$. (Of course, we are now interested in its value at 0.) Then, the analogue of Theorem 3.1 is the following result, which is basically proved as before.

**Theorem 3.11.** *Let $\Gamma$ be a graph with Laplacian matrix eigenvalues $\mu_0 = 0 < \mu_1 < \cdots < \mu_d$, and let $Q_k$ be the polynomial defined as above. Then,*

$$Q_k(0) > \sqrt{\left(\frac{n}{s} - 1\right)\left(\frac{n}{t} - 1\right)} \Rightarrow D_{(s,t)}(\Gamma) \leqslant k. \tag{15}$$

If $\Gamma$ is a $\delta$-regular graph and has adjacency matrix eigenvalues $\lambda_0 (= \delta), \lambda_1, \ldots, \lambda_d$, then its Laplacian eigenvalues are $\mu_l = \delta - \lambda_l$, $0 \leqslant l \leqslant d$. Since the maximum value attained by the alternating polynomials only depends on the relative position of the points, we clearly have $Q_k(0) = P_k(\lambda)$, and the above theorem also leads to Corollary 3.5(a). In other words, we can say that, for regular graphs, Theorems 3.1 and 3.11 are equivalent.

The corresponding analogue of Corollary 3.3 can also be deduced from Theorem 3.11. Notice that now $\pi_l = \prod_{j=0(j \neq l)}^{d} |\mu_l - \mu_j|$ $(0 \leqslant l \leqslant d)$.

In fact, following the ideas in Chung et al. [5], the proof of Theorem 3.11 also works if $\Gamma$ is a *locally regular* directed graph, that is $\delta_i^+ = \delta_i^-$ for any $e_i \in V$. Indeed, in this case the Laplacian matrix can also be defined and, although its eigenvalues are not necessarily real, it still has $\mu_0 = 0$ with associated eigenvector $\boldsymbol{u}$ satisfying $\boldsymbol{u}^t\boldsymbol{Q} = \boldsymbol{Q}\boldsymbol{u} = \boldsymbol{0}$ (a property which is needed in the proof). Moreover, by the theorem of Gersgorin, all the other eigenvalues are in the circle with centre and radius $\Delta = \max_{1 \leqslant i \leqslant n} \delta_i$. Then, we can take as $Q_k$ the corresponding complex alternating polynomial, as defined in Section 2, and derive some corollaries similar to the case of graphs.

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